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Dynamics of Rational Symplectic Mappings and Difference Galois Theory

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In this paper, we study the relationship between the integrability of rational symplectic maps and difference Galois theory. We present a Galoisian condition, of Morales–Ramis type, ensuring the nonintegrability of a rational symplectic map in the noncommutative sense (Mishchenko–Fomenko). As a particular case, we obtain a complete discrete analogue of Morales–Ramis Theorems for nonintegrability in the sense of Liouville.

1 Introduction and Organization

The problem of deciding whether a given continuous dynamical system is integrable is an old and difficult problem. Many examples arise in classical mechanics and physics, the *three body problem* being one of the most famous. In 1999, after anterior works by a number of authors among whom Kowalevskaya, Poincaré, Painlevé and more recently

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Ziglin [25], Morales and Ramis exhibited in [13–15] an algebraic obstruction to the integrability of Hamiltonian dynamical systems. This obstruction lies in the differential Galois group of the linearization of the dynamical system under consideration along a given particular solution (the variational equation). During the last years, the method developed by Morales and Ramis have shown its efficiency, permitting the resolution of numerous classical problems.

Similar questions occur in the study of discrete dynamical systems. For instance, a rational map $f : \mathbb{P}^2(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C})$ being given the typical question is: does there exist a nontrivial rational invariant of f? That is, does there exist $H \in \mathbb{C}(x, y)$ nonconstant such that $H \circ f = H$? In case of a positive answer, f is called integrable (we refer to Section 3 for the general notion of integrability). Inspired by Morales–Ramis theory, we point out an algebraic criterion of nonintegrability based on difference Galois theory. Our method involves the linearization of the dynamical system under consideration along a "particular solution" (that is an invariant parameterized complex algebraic curve of genus zero $\iota(\mathbb{P}^1(\mathbb{C}))$) called the discrete variational equation. From a technical point of view, we present two proofs of our main theorem: the first one is based on the difference Galois theory developed by Singer and van der Put in [23] (there are some complications with respect to the differential case due to the fact that the Picard–Vessiot ring associated to a given difference equation is not necessarily a domain), whereas the second one is based on Malgrange theory.

The paper is organized as follows. In Sections 2 and 3, definitions and notations used in the article are given. In Section 4, we introduce the Galois groupoid of Malgrange and prove that for an integrable system the groupoid of Malgrange is infinitesimally commutative. But commutativity of the Malgrange groupoid is not easily checkable. In Section 5, we recall some facts about difference equations from [23]. Section 6 deals with linearization of the discrete system and its differential invariants along an invariant curve. The main theorem is proved in Section 7:

Theorem 1.1. Let f be an integrable symplectic map having a ϕ-adapted curve ι. Then the neutral component of the Galois group of the discrete variational equation of f along ι is commutative. □

The two proofs are given in this section. The first one deals directly with the linearized system following the original approach of Morales–Ramis [14]. The second used the linearization to check the commutativity of Malgrange groupoid. This work is based on a suggestion of Ramis; we are pleased to thank him.

2 Notations

Unless explicitly stated otherwise, in the whole article we will use the following notations and conventions:

- the (symplectic) manifolds and mappings are complex;
- V will denote a complex algebraic variety of even complex dimension $2n$ ($n \in \mathbb{N}^*$);
- $\mathbb{C}(V)$ will denote the field of rational functions over the algebraic variety V;
- ω will denote a nondegenerate closed holomorphic 2-form on V (symplectic form);
- $f : V \dashrightarrow V$ will denote a birational mapping preserving ω that is such that $f^*\omega = \omega$ (symplectic mapping of (V, ω));
- the symplectic gradient of a rational function $H : V \dashrightarrow \mathbb{C}$ is the rational vector field X_H satisfying $dH = \omega(X_H, \cdot)$;
- the induced Poisson bracket on $\mathbb{C}(V)$ will be denoted by $\{H_1, H_2\} = \omega(X_{H_1}, X_{H_2})$.

3 Integrability of a Symplectic Map

For the sake of completeness, we first recall the notion of Liouville integrability of a continuous hamiltonian dynamical system.

Definition 3.1 [9]**.** Let H be a rational Hamiltonian on V. It is said to be integrable in the Liouville sense (or in the Langrangian way) if there is $H_1, \ldots, H_n$ functionally independent first integrals of H (*i.e.* $\{H, H_j\} = 0$) in involution:

- functionally independent means that their differentials $dH_1(z), \ldots, dH_n(z)$ are linearly independent for at least one point $z \in V$ or, equivalently, for all z in a Zariski dense open subset of V;
- in involution means that, for all i, j in $[1, n]$, $\{H_i, H_j\} = 0$. □

This means that the motion takes place in the fibers of a Lagrangian fibration. This notion of integrability was generalized by Fomenko–Mishchenko [5] (see also [1, 17]) following the work of Marsden–Weinstein [12].

Definition 3.2. Let H be a rational Hamiltonian on V. It is said to be integrable in the noncommutative sense (or in the isotropic way) if there is $H_1, \ldots, H_{n+\ell}$ independent first integrals of H, *i.e.* $\{H, H_j\} = 0$ such that the distribution $\bigcap_i \ker dH_i$ is generated by the symplectic gradients $X_{H_1}, \dots, X_{H_{n-\ell}}$ as vector space over $\mathbb{C}(V)$. □

This means that the motion takes place in the fibers of an isotropic fibration. Mishchenko and Fomenko conjectured that this notion of integrability is equivalent to Liouville's one. This question will not be discussed here. For a proof in the finite-dimensional case, we refer the reader to [2, 21]. A Galoisian condition for integrability in the isotropic way was obtained by Maciejewki and Przybylska in [10]. As an evidence of Mishchenko–Fomenko conjecture, the Galoisian conditions in both cases are the same.

In the context of discrete dynamical systems (iteration of maps) the same notion of integrability is used.

Definition 3.3. The symplectic map f is integrable (in the isotropic way) if it admits $n+\ell$ first integrals $H_1, \dots, H_{n+\ell}$ in $\mathbb{C}(V)$, functionally independent such that $\bigcap_i \ker dH_i$ is generated by the symplectic gradients $X_{H_1}, \dots, X_{H_{n-\ell}}$. □

The only point in this definition requiring some comments is the notion of first integral. A rational function $H \in \mathbb{C}(V)$ is a first integral of f if H is constant over the trajectories of f, that is $H \circ f = H$.

4 A Nonlinear Approach

4.1 Malgrange groupoid and its Lie algebra

Let V be a dimension n smooth complex algebraic variety. Let $R_q V$ denote the order q frame bundle on V. This bundle is the space of order q jets of invertible maps $r : (\mathbb{C}^n, 0) \to V$. Such a jet will be denoted by $j_q(r)$.

As soon as one gets local coordinates $r_1, \dots, r_n$ on an open subset U of V, one gets local coordinates r_i^α with $i \in [1, \dots, n]$ and $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq q$ on the open set of $R_q V$ of order q frames with target in U. This bundle is an algebraic variety and is naturally endowed with two algebraic actions. The first one is the action of the algebraic group Γ_q^n of order q jets of invertible maps $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ by composition at the source. For this action $R_q V$ is a Γ_q^n-principal bundle on V.

The second action is the action of the algebraic *groupoid* $J_q^*(V, V)$ of invertible local holomorphic maps $s : (V, a) \to (V, b)$ by composition at the target.

Let us recall the definition of an algebraic groupoid on V.

Definition 4.1. An algebraic groupoid on V is an algebraic variety G with:

- two maps $s, t : G \to V$, the *source* and the *target*,
- an associative composition $\circ : G_V {}_s\!\times_t G \to G$,
- an identity $id : V \to G$ such that for all $g \in G$

$$g \circ id(s(g)) = id(t(g)) \circ g = g,$$

- an inverse ${}^{\circ -1} : V \to V$ such that $\forall g \in G$

$$s(g^{\circ-1}) = t(g), \quad t(g^{\circ-1}) = s(g), \quad g \circ g^{\circ-1} = id(t(g)) \quad \text{and} \quad g^{\circ-1} \circ g = id(s(g)).$$

The induced morphisms on the structure (sheaves of) rings must satisfy the dual diagrams. □

The varieties $J_q^*(V, V)$ are archetypes of groupoid on V.

Definition 4.2. Let $f : V \dashrightarrow V$ be a rational map. The order q prolongation of f is $R_q f : R_q V \dashrightarrow R_q V$ defined by $(R_q f)(j_q(r)) = j_q(f \circ r)$. □

The basic objects used to understand the dynamic of f are (rational) first integrals, also called (rational) invariants, i.e. functions $H : V \dashrightarrow \mathbb{C}$ such that $H \circ f = H$. From a differential-algebraic point of view, invariants of f are replaced by differential invariants of f.

Definition 4.3. A differential invariant of order q of f is an invariant of $R_q f$, *i.e.* $H : R_q V \dashrightarrow \mathbb{C}$ such that $H \circ R_q f = H$. □

Thanks to the canonical projections $\pi_q^{q+1} : R_{q+1} V \to R_q V$, an order q invariant is also an order $q+1$ invariant. Let us denote by $\mathcal{D}_q Inv(f)$ the set of all the differential invariants of order q of f. This is a $\mathbb{C}$-algebra of finite type. Invariant tensor fields are particular cases of order 1 differential invariants.

Definition 4.4. Let V be a smooth complex algebraic variety and $f : V \dashrightarrow V$ be a rational map. The $\mathcal{D}_q$-closure of f, denoted by $G_q(f)$, is the subvariety of $J_q^*(V, V)$ defined by the equations $H \circ j_q(s) = H$ for all $H \in \mathcal{D}_q Inv(f)$. □

Remark. In general, these subvarieties are not subgroupoids but subgroupoids above a Zariski dense open subset of V. □

Let $J^*(V, V)$ be the projective limit $\lim\limits_{\leftarrow} J_q(V, V)$ and π_q be the projection of this space on $J_q(V, V)$.

Definition 4.5 (Malgrange). Let V be a smooth complex algebraic variety and $f : V \dashrightarrow V$ be a rational map. The Malgrange groupoid of f is the subvariety of $J^*(V, V)$ given by $\bigcap \pi_q^{-1} G_q(\Psi)$ and it is denoted by $G(f)$. □

Remark. This is not the original definition from Malgrange [11], which is about foliation. The extension of the definition to discrete dynamical systems is straightforward. Proposition 2.36 from Chapter 3 of [18] (pp. 467–9) allows us to "simplify" the exposition by using directly differential invariants. □

A solution of $\bigcap \pi_q^{-1} G_q(\Psi)$ is a formal map $s : \widehat{V, a} \to \widehat{V, b}$ whose coefficients satisfy all the equations of the $G_q(\Psi)$. It is said to be convergent if the formal map converges. There are well-known solutions of $G(f)$: the iterates $f^{\circ n}$ preserve all the differential invariants of f.

For a rational map $\Phi : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, the Malgrange groupoid is computed in [4]. Following this computation the maps with a "small" Malgrange groupoid are the maps called integrable, i.e. maps with a nontrivial commutant [3, 19, 24].

Groupoids are not amenable objets. It is much easier to work with the infinitesimal part of a groupoid: its (analytic sheaf of) Lie algebra. For subgroupoids of $J^*(V, V)$ these Lie algebras are algebras of vector fields. Let us give some notations before the definitions. Let $r_1, \ldots, r_n$ be local coordinates on V and r_i^α for $1 \leq i \leq n$ and $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq q$ be the induced coordinates. A function on $R_q V$ can be differentiated with respect to coordinates on $\mathbb{C}^n$ by mean of the chain rule, i.e. using the vectors fields

$$D_j = \sum_{i,\alpha} r_i^{\alpha+\epsilon(j)} \frac{\partial}{\partial r_i^\alpha},$$

where $\epsilon(j) = (0 \cdots 0, \overset{\overset{j}{\downarrow}}{1}, 0 \cdots 0)$, one gets a function on $R_{q+1} V$.

Definition 4.6. Let $X = \sum_i a_i \frac{\partial}{\partial r_i}$ be a holomorphic vector field on an open subset $U \subset V$. The order q prolongation of X is the vector field $R_q X = \sum_{i,\alpha} D^\alpha a_i \frac{\partial}{\partial r_i^\alpha}$ on $R_q U \subset R_q V$. □

Definition 4.7. Let V be a smooth complex algebraic variety and $f : V \dashrightarrow V$ be a rational map. The Lie algebra of the Malgrange groupoid of f is the analytic sheaf of vector field X on V such that

$$\forall q \in \mathbb{N} \text{ and } \forall H \in \mathcal{D}_q Inv(f), \quad (R_q X)H = 0.$$

□

Remark. The original definition from [11] is richer about the structure of this sheaf but for the sequel this definition will be sufficient. □

4.2 Malgrange groupoid of an integrable symplectic map

As we already said invariant tensors are examples of order 1 differential invariants. We will describe these invariants and their lifting up in some cases.

Lemma 4.1. Let T be a rational tensor field on $R_q V$. It induces m functions h_T^i, $1 \le i \le m$, on $R_{q+1} V$. Furthermore, for a vector field X on V one gets

$$L_{R_q X} T = 0 \text{ if and only if } (R_{q+1} X) h_T^i = 0 \qquad \forall 1 \le i \le m.$$

□

Let us see examples of these functions and this property.

Rational function: A rational function $h : V \dashrightarrow \mathbb{C}$ can be lifted to a rational function on $R_q V$. It is clear that if X is a vector field on V and h is X-invariant then its lifting up is $R_q X$-invariant.

Rational form: By the very definition, if ω is a p-form on V then it is a function on $(TV)^p = J_1((\mathbb{C}^p, 0) \to V)$. So it induces $\binom{n}{p}$ functions on $R_1 V$. One can then lift these functions on higher order frame bundles.

Now if X is a vector field on V and ω is a X-invariant p-form. Let $r_1, \ldots, r_n$ be local coordinates on V, one gets

$$\omega = \sum_{I \in C(p,n)} w_I dx_{\wedge I} \text{ and } X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial r_i},$$

where $C(p,n)$ stands for the set of length p increasing sequences $i_1 < \cdots < i_p$ in $\{1,\ldots,n\}$ and $dr_{\wedge I}$ stands for $dr_{i_1} \wedge \cdots \wedge dr_{i_p}$. The lifting up of ω to $R_1 V$ is the $\binom{n}{p}$-uple of functions:

$$h^J_\omega\big(- r_i - r_i^{\epsilon(j)} - \big) = \sum w_I r^J_I,$$

where $J \subset \{1,\ldots,n\}$ with $\#J = p$ and r^J_I stands for the determinant of the matrix $(r_i^{\epsilon(j)})_{\substack{i\in I\\ j\in J}}$.

Easy computations give

$$\begin{aligned} L_X\omega &= \sum X w_I dr\wedge I + \sum (-1)^{\ell+1} w_I \frac{\partial a_p}{\partial r_k} dr_k \wedge dr_{\wedge I - \{i_\ell\}}, \\ R_1 X &= \sum a_i \frac{\partial}{\partial r_i} + \sum \frac{\partial a_i}{\partial r_k} r_k^{\epsilon(j)} \frac{\partial}{\partial r_i^{\epsilon(j)}}, \\ \text{and} \quad (R_1 X)(h^J_\omega) &= \sum X w_I r^J_I + \sum w_I \frac{\partial a_i}{\partial r_k} r_k^{\epsilon(j)} \frac{\partial r^J_I}{\partial r_i^{\epsilon(j)}}. \end{aligned}$$

Thus one has $h^J_{L_X\omega} = (R_1 X)(h^J_\omega)$.

For instance, starting from a X-invariant function h on V, one gets $n+1$ order 1 differential invariants: h itself and $h^1_{dh}, \ldots, h^n_{dh}$.

Rational vector field: Let v be a rational vector field on $V : v = \sum v_i \frac{\partial}{\partial r_i}$ in local coordinates. An order 1 frame determines a basis $e_1, \ldots, e_n$ of $T_r V$. The coordinates of $v(r)$ in this basis determine n functions $h^i_v : R_1 V \dashrightarrow \mathbb{C}$.

Higher order forms: A p-form ω on $R_q V$ determines functions on $R_1 R_q V$. The induced function on $R_{q+1} V$ are the pull-back by the natural inclusion $R_{q+1} V \subset R_1 R_q V$.

Theorem 4.1. The Lie algebra of Malgrange groupoid of an integrable (in the isotropic way) symplectic map is commutative. □

Proof. Vector fields of the Lie algebra of Malgrange groupoid must preserve $n+\ell$ independent functions $H_1, \ldots, H_{n+\ell}$: $XH_i = 0$. Because they preserve H_i and the symplectic form ω, they must preserve the symplectic gradients X_i.

On the subvarieties $H_i = c_i$ the integrable map preserves a parallelism given by $X_{H_1}, \ldots, X_{H_{n-\ell}}$. This parallelism is commutative: $[X_{H_i}, X_{H_j}] = X_{H_j} H_i = 0$ for all $1 \le i \le n+\ell$ and $1 \le j \le n-\ell$. Because the Lie algebra of vector fields preserving a commutative parallelism is commutative (see for instance [8]), the theorem is proved. ■

For Liouville integrable Hamiltonian vector fields, this theorem was already proved along the same lines by Ramis [16].

5 Linear Difference Equations

5.1 Basic concepts

For the convenience of the reader, we recall some well-known concepts from difference equations theory.

Definition 5.1. Let ϕ be a nonperiodic Moebius transformation of $\mathbb{P}^1(\mathbb{C})$ and $E \to \mathbb{P}^1(\mathbb{C})$ a rank n vector bundle on the projective line. A rank n ϕ-difference system on E is a lift $\Phi : E \to E$ of ϕ linear on the fibers. Let U be a ϕ-invariant open set of $\mathbb{P}^1(\mathbb{C})$ for the transcendent topology. A section $Y : U \to E$ is a solution of Φ if its graph is Φ-invariant. □

Using a rational change of variable any Moebius transformation can be converted either into a translation $\phi : z \mapsto z + h$ with $h \in \mathbb{C}$ or into a q-dilatation $\phi : z \mapsto qz$ with $q \in \mathbb{C}^*$, depending on the number of fixed points of ϕ on $\mathbb{P}^1(\mathbb{C})$ (one in the first case, two in the second case). The corresponding equations are respectively called finite difference and q-difference equations.

By a gauge transformation, any ϕ-difference system can be transformed in a system on a trivial vector bundle. In this case, considering a coordinate z on the projective line, Φ is described by an invertible matrix with rational entries:

$$\begin{aligned} \Phi : \mathbb{P}^1(\mathbb{C}) \times \mathbb{C}^n &\to \mathbb{P}^1(\mathbb{C}) \times \mathbb{C}^n \\ (z, Y) &\mapsto (\phi z, A(z)Y) \end{aligned}$$

with $A \in GL_n(\mathbb{C}(z))$. The equations of invariance of a solution Y are:

$$Y(\phi z) = A(z)Y(z). \tag{1}$$

In the sequel, vector bundles will be endowed with a fiberwise symplectic form ω and Φ will preserve this form. By a fiberwise linear change of variables, one can assume that ω is the canonical symplectic J form on fibers. Such a ϕ-difference system has a matrix A in $Sp_{2n}(\mathbb{C}(z))$.

What precedes can be rephrased and generalized using the language of difference modules; these notions being used in [23], one of our main references for the Galois theory of difference equations with [22], we shall remind them. A *difference ring* is a couple (R, ϕ) where R is a commutative ring and where ϕ is a *difference operator*, that is an automorphism of R. If $R = k$ is a field, then (k, ϕ) is called a *difference field*. A *difference module M* over a difference field (k, ϕ) is a $k[\phi, \phi^{-1}]$-module, whose k-algebra obtained after restriction of scalars is finite dimensional. The choice of a particular k-basis $\mathcal{E}$ of M provides a difference system with coefficients in k : $\phi Y = AY$, where $A \in Gl_n(k)$ is the matrix representing $\phi\mathcal{E}$ in $\mathcal{E}$; the difference system obtained by choosing another basis $\mathcal{E}'$ takes the form $\phi Y = A'Y$ with $A' = (\phi P)^{-1}AP$ and $P \in Gl_n(k)$. Conversely, starting from a difference system $\phi Y = AY$, $A \in Gl_n(k)$, we construct a difference module in an obvious way; the difference module obtained from $\phi Y = A'Y$ with $A' = (\phi P)^{-1}AP$ and $P \in Gl_n(k)$ is isomorphic to those obtained from A.

5.2 Picard-Vessiot theory for linear difference equations

In the whole section, we consider a difference field (k, ϕ) with algebraically closed field of constants (we remind that the field of constants of (k, ϕ) is $C = \{c \in k \mid \phi c = c\}$) and:

$$\phi Y = AY, \tag{2}$$

a difference system with coefficients in k (i.e. $A \in Gl_n(k)$).

We shall remind the construction of the difference Galois group of (2) due to Van der Put and Singer in [23].

The counterpart for difference equations of the field of decomposition of an algebraic equation is the Picard–Vessiot extension.

Definition 5.2. A Picard–Vessiot ring for the difference system (2) is a k-algebra R such that:

(i) an automorphism of R, also denoted by ϕ, which extend ϕ is given;
(ii) R is a simple difference ring;
(iii) there exists a fundamental matrix $\mathfrak{Y}$ for $\phi Y = AY$ having entries in R such that $R = k[(\mathfrak{Y})_{i,j})_{1\le i,j\le n}, \det(\mathfrak{Y})^{-1}]$. □

Let us precise the terminology used in the previous definition:

- a difference ideal of a difference ring is an ideal $\mathcal{I}$ stable by ϕ. We say that R is a simple difference ring if it has only trivial difference ideal;
- a fundamental matrix $\mathfrak{Y}$ for $\phi Y = AY$ having entries in R is a matrix $\mathfrak{Y} \in Gl_n(R)$ such that $\phi\mathfrak{Y} = A\mathfrak{Y}$.

Van der Put and Singer proved in [23] that such an extension exists and that it is unique up to isomorphism of difference rings. Furthermore, the field of constants of the Picard–Vessiot extension is equal to the fields of constants of the base field.

Let us give the general lines of the proof of the existence of the Picard–Vessiot extension. We denote by $X = (X_{i,j})_{1\le i,j\le n}$ a matrix of indeterminates over k and we extend the difference operator ϕ to the k-algebra

$$U = k\left[(X_{i,j})_{1\le i,j\le n}, \det(X)^{-1}\right]$$

by setting $(\phi X_{i,j})_{1\le i,j\le n} = A(X_{i,j})_{1\le i,j\le n}$. Then, for any maximal difference ideal $\mathcal{I}$ (which is not necessarily a maximal ideal) of U, the couple $(R = U/\mathcal{I}, \phi)$ is a Picard–Vessiot ring for the difference system (2).

Definition 5.3. The Galois group G of the difference system (2) is the group of difference automorphisms of its Picard–Vessiot extension over the base field (k, ϕ). It is a linear algebraic group over the field of constants C. □

If A "takes its values" in a particular linear algebraic group, then the Galois group of (2) is a subgroup of this algebraic group. For later use, let us reprove this statement in the symplectic case.

Lemma 5.1. Assume that $k = \mathbb{C}(z)$ and that $A \in Gl_n(\mathbb{C}(z))$ is sympletic: ${}^tAJA = Id$ where J denotes the matrix of the symplectic form. Then there exists R a Picard–Vessiot extension of (2) containing a symplectic fundamental system of solutions of $\phi Y = AY$; that is, there exists $\mathfrak{Y}$ a fundamental system of solutions of $\phi Y = AY$ with entries in R such that ${}^t\mathfrak{Y}J\mathfrak{Y} = Id$. The corresponding Galois group belongs to the symplectic group. □

Proof. As above, we denote by $X = (X_{i,j})_{1\le i,j\le n}$ a matrix of indeterminates over k and we extend the difference operator ϕ to the k-algebra

$$U = k[(X_{i,j})_{1\le i,j\le n}, \det(X)^{-1}]$$

by setting $(\phi X_{i,j})_{1\le i,j\le n} = A(X_{i,j})_{1\le i,j\le n}$. Remark that the ideal $\widetilde{\mathcal{I}}$ of U generated by entries of ${}^tXJX - Id$ is a difference ideal. Indeed this ideal is made of the linear combinations

$\sum_{i,j} a_{i,j}({}^t XJX - Id)_{i,j}$ and

$$\begin{aligned}\phi \textstyle\sum_{i,j} a_{i,j}({}^t XJX - Id)_{i,j} &= \textstyle\sum_{i,j} \phi(a_{i,j})({}^t \phi(X)J\phi(X) - Id)_{i,j}\\ &= \textstyle\sum_{i,j} \phi(a_{i,j})({}^t (AX)J(AX) - Id)_{i,j}\\ &= \textstyle\sum_{i,j} \phi(a_{i,j})({}^t XJX - Id)_{i,j}.\end{aligned}$$

Hence, there exists a maximal difference ideal $\mathcal{I}$ of R containing $\widetilde{\mathcal{I}}$, so that $U/\mathcal{I}$ is a Picard–Vessiot ring for $\phi Y = AY$. The lemma follows easily. ∎

We come back to the general non-necessarily symplectic situation. In contrast with the differential case, the Picard–Vessiot ring is not necessarily a domain. Let us go further into the study of its structure. Following [23], the Picard–Vessiot ring, say R, can be decomposed as a product of domains: $R = R_1 \oplus \cdots \oplus R_s$ such that

(i) $\phi R_i = R_{i+1}$;

(ii) each component R_i is a Picard–Vessiot ring for the difference equation

$$\phi^s Y = (\phi^{s-1} A \cdots A)Y, \tag{3}$$

the corresponding Galois group is denoted by G'.

The proof of the following lemma is left to the reader.

Lemma 5.2. If the difference system (2) is integrable then it is also the case of the difference system (3). □

Furthermore, following [23], we have a short exact sequence:

$$0 \to G' \to G \to \mathbb{Z}/s\mathbb{Z} \to 0. \tag{4}$$

The following result will be essential in what follows.

Lemma 5.3. If G'^0 is abelian, then G^0 is abelian. □

Proof. Indeed, the above exact sequence (4) allows us to identify G' with an algebraic subgroup of G of finite index, so that $G^0 \subset G'^0$. ∎

6 Linearization of Integrable Dynamical Systems

In this section, starting with a discrete dynamical system f having an invariant curve of genus zero on which f is a Moebius transform, we explain how one can obtain by a linearization procedure a ϕ-difference system (*finite difference or q-difference system*). Differential invariants of f provide in some cases differential invariants for its linearization.

6.1 Adapted curves and discrete variational equations

Definition 6.1. A curve ϕ-adapted to f is a rational embedding $\iota : \mathbb{P}^1(\mathbb{C}) \dashrightarrow V$ such that $f \circ \iota = \iota \circ \phi$. □

Definition 6.2. Let ι be a curve ϕ-adapted to f. The difference system $\phi Y = Df(\iota) Y$ over $\mathbb{P}^1(\mathbb{C})$ is called the discrete variational equation of f along ι. □

By a rational gauge transform, the pull-back of the tangent bundle : $\iota^* TV$ can be assume to be trivial and the variational equation can be written as a ϕ-difference system on the projective line as stated in the definition.

6.2 Integrability of the discrete variational equation

In this section, we prove that the discrete variational equation of f along a ϕ-adapted curve is integrable in the following sense:

Definition 6.3. A rank $2n$ difference equation $\phi Y = AY$, $A \in SL_{2n}(\mathbb{C}(z))$ is integrable if it admits $n + \ell$ independent rational first integrals $h_1, \dots, h_{n+\ell}$ such that the distribution $\ker dz \cap \bigcap_i \ker dh_i$ is generated by the fiberwise symplectic gradients $X_{h_1}, \dots, X_{h_{n-\ell}}$. □

There are two points requiring comments:

- A function $H \in \mathbb{C}(z, Y_i | 1 \leq i \leq 2n)$ is a first integral of a difference system $\phi Y = AY$ if $H(\phi z, AY) = H(z, Y)$.
- The symplectic structure on the fibers is given by the constant canonical symplectic structure on $\mathbb{C}^{2n}$.

Our main tool is the *generic junior part* that we shall now introduce.

Let us consider a rational embedding $\iota : \mathbb{P}^1(\mathbb{C}) \dashrightarrow V$. The set of rational functions on V, whose polar locus does not contain $\iota(\mathbb{P}^1(\mathbb{C}))$, is denoted by $\mathbb{C}[V]_\iota$.

Definition 6.4. The generic valuation $\nu_\iota(H)$ of a function $H \in \mathbb{C}[V]_\iota$ along ι is defined by:

$$\nu_\iota(H) = \min\{k \in \mathbb{N} \mid D^k H(\iota) \not\equiv 0 \text{ over } \mathbb{P}^1(\mathbb{C})\}.$$

It extends to $H \in \mathbb{C}(V)$ by setting $\nu_\iota(H) = \nu_\iota(F) - \nu_\iota(G)$, where $H = F/G$ with $F, G \in \mathbb{C}[V]_\iota$. □

Definition 6.5. Let us consider $H \in \mathbb{C}[V]_\iota$ and $T_\iota = \iota^* TV$. We define $H_\iota^\circ : T_\iota \dashrightarrow \mathbb{C}$, the generic junior part of H along ι, by

$$H_\iota^\circ = D^{\nu_\iota(H)} F(\iota).$$

We extend this definition to $H \in \mathbb{C}(V)$ by setting $H^\circ = F^\circ / G^\circ$ for any $F, G \in \mathbb{C}[V]_\iota$ such that $H = F/G$. □

The reader will easily verify that the above definitions do not depend on a particular choice of F, G.

Remark. Junior parts of F at points $p \in \iota(\mathbb{P}^1(\mathbb{C}))$ in the sense of [13] coincide with the generic junior part over a Zariski-dense subset of $\mathbb{P}^1(\mathbb{C})$. □

The interest of the generic junior part in our context consists in the fact that it converts a first integral of f into a first integral of the variational equation.

Lemma 6.1. Let us consider $H \in \mathbb{C}(V)$ a first integral of f. Then H_ι° is a first integral of the variational equation of f along ι. □

Proof. Let us consider $F, G \in \mathbb{C}[V]_\iota$ such that $H = F/G$. The assertion follows by differentiating $\upsilon_\iota(F) + \upsilon_\iota(G)$ times the equality $(F \circ f) \cdot G = (G \circ f) \cdot F$. ■

Therefore, if $H_1, \dots, H_{n+\ell}$ are $n + \ell$ first integrals of f, then $(H_1)_\iota^\circ, \dots, (H_{n+\ell})_\iota^\circ$ are $n + \ell$ first integrals of the variational equation of f along ι. However, in general, these function are *not* functionally independent, even if $H_1, \dots, H_{n+\ell}$ are functionally independent. This difficulty can be overcome by using the following lemma due to Ziglin [25].

Lemma 6.2. Let $F_1, \dots, F_k$ be k meromophic functions functionally independent in a neighborhood of $\iota(\mathbb{P}^1(\mathbb{C}))$. Then there exist k polynomials $P_1, \dots, P_k \in \mathbb{C}[z_1, \dots, z_k]$ such that the generic junior part $(G_1)^\circ_\iota, \dots, (G_k)^\circ_\iota$ along ι of $G_1 = P_1(F_1, \dots, F_k), \dots, G_k = P_k(F_1, \dots, F_k)$ are functionally independent. □

Proof. The usual Ziglin lemma ensures that the assertion is true if one replaces "generic junior part" by "junior part at a given point." The previous remark allows us to conclude the proof. ■

Theorem 6.1. The variational equation of f along a ϕ-adapted curve φ is integrable. □

Proof. Let $H_1, \dots, H_{n+\ell}$ be first integrals of f with symplectic gradients $X_{H_1}, \dots, X_{H_{n-\ell}}$ generating the distribution $\bigcap_i \ker dH_i$.

By Ziglin lemma, junior parts of the first $n-\ell$ first integrals give $n-\ell$ independent first integrals $(H_1)^\circ_\iota, \dots, (H_{n-\ell})^\circ_\iota$ for the variational equation. The independence of the differentials $d(H_1)^\circ_\iota, \dots, d(H_{n-\ell})^\circ_\iota$ implies the independence of their symplectic gradients $X_{(H_1)^\circ_\iota}, \dots, X_{(H_{n-\ell})^\circ_\iota}$. The set of first integrals is completed applying Ziglin lemma to the 2ℓ remaining first integrals. ■

In the next sections, we will prove that the integrability of the variational equation implies severe constraints on its Galois group (more precisely on the neutral component of the Galois group).

7 Integrability and Galois Theory

7.1 Statement of the main theorem

Theorem 7.1. Let f be an integrable symplectic map having a ϕ-adapted curve ι. Then the neutral component of the difference Galois group of the discrete variational equation of f along ι is commutative. □

In the following sections, we give two proofs of our main theorem: the first one is based on Picard–Vessiot theory developed by Van der Put and Singer; the second one is based on Malgrange groupoid.

7.2 First proof of the main theorem: Picard–Vessiot approach

Lemma 7.1. Let $(k = \mathbb{C}(z), \phi)$ be a difference field with field of constants $\mathbb{C}$. Let H be a first integral of the difference system with coefficients in $\mathbb{C}(z) : \phi Y = AY$. Let R be a Picard–Vessiot extension for this difference system, which is supposed to be a domain. Let $\mathfrak{Y} \in Gl_n(R)$ be a fundamental system of solutions with coefficients in R. Then the function:

$$
\begin{aligned}
{}_{\mathfrak{Y}}H : \mathbb{C}^n &\to \mathbb{C} \\
y &\mapsto H(z, \mathfrak{Y}y)
\end{aligned}
$$

□

is invariant under the action of the Galois group G.

The action of the Galois group is given by the following: for $(\sigma, y) \in G \times \mathbb{C}^n$, we set $\sigma \cdot y = C(\sigma)y$ where $C(\sigma) \in Gl_n(\mathbb{C})$ is such that $\sigma\mathfrak{Y} = \mathfrak{Y}C(\sigma)$.

Proof. The first point to check is that, for any $y \in \mathbb{C}^n$, $H(z, \mathfrak{Y}y) \in \mathbb{C}$. *A priori* $H(z, \mathfrak{Y}y)$ belongs to Frac(R), the Picard–Vessiot field, but $\phi H(z, \mathfrak{Y}y) = H(\phi z, (\phi\mathfrak{Y})y) = H(\phi z, (A(z)\mathfrak{Y})y) = H(\phi z, A(z)(\mathfrak{Y}y)) = H(z, \mathfrak{Y}y)$ so that $H(z, \mathfrak{Y}y)$ is ϕ-invariant: $H(z, \mathfrak{Y}y)$ belongs to the field of contant of the Picard–Vessiot field, that is to $\mathbb{C}$. Since $H(z, \mathfrak{Y}y)$ lies in the field of constants $\mathbb{C}$, it is invariant by the Galois group: $\sigma H(z, \mathfrak{Y}y) = H(z, \mathfrak{Y}y)$. But, since H has rational coefficients in z and that $\mathbb{C}(z)$ is invariant by the action of the Galois group, we also have, for all $\sigma \in G$, $\sigma H(z, \mathfrak{Y}y) = H(z, \sigma\mathfrak{Y}y) = H(z, \mathfrak{Y}C(\sigma)y) = H(z, \mathfrak{Y}(\sigma \cdot y))$. ■

Lemma 7.2. Suppose that the symplectic difference equation $\phi Y = AY$ with coefficients in $\mathbb{C}(z)$ is integrable and that its Picard-Vessiot ring is a domain. Then the Lie algebra of its Galois group is abelian. □

Proof. Let $H_1, \dots, H_{n+\ell}$ be $n + \ell$ first integrals insuring integrability of the difference equation as in Definition 6.3. Lemma 5.1 ensures that there exists a Picard–Vessiot extension R containing a symplectic fundamental system of solution $\mathfrak{Y} \in \mathrm{Sp}_{2n}(R)$. Because

$$
\sum_j \frac{\partial\, {}_{\mathfrak{Y}}H_i}{\partial y_j} dy_j = \sum_{j,k} \frac{\partial H_i}{\partial Y_k} \mathfrak{Y}_{k,j} dy_j
$$

and invertibility of $\mathfrak{Y}$ independence of the $n + \ell$ forms $\sum_k \frac{\partial H_i}{\partial Y_k} dY_k$ implies independence the $n + \ell$ differentials $d\, {}_{\mathfrak{Y}}H_i$. Moreover, from Lemma 7.1, these functions are invariant under the action of the Galois group.

Let X be a linear vector field on $\mathbb{C}^{2n}$ in the Lie algebra of the Galois group.

By the assumption of integrability, all vector field preserving ${}_{\mathfrak{Y}}H_1, \ldots, {}_{\mathfrak{Y}}H_{n+\ell}$ are combinations of the $n-\ell$ first symplectic gradients:

$$X = \sum_{i=1}^{n-\ell} c_i X_{{}_{\mathfrak{Y}}H_i}$$

with $c_i \in \mathbb{C}(y_1, \ldots, y_{2n})$.

Because the Galois group is a subgroup of $\mathrm{Sp}_{2n}(\mathbb{C})$, if it preserves a function, it preserves its symplectic gradient too. Thus one gets $[X_{{}_{\mathfrak{Y}}H_j}, X] = 0$ for all $1 \le j \le n+\ell$ and

$$dc_i = \sum_{i=1}^{n+\ell} c_i^j \, d\,{}_{\mathfrak{Y}}H_j$$

for some $c_i^j \in \mathbb{C}(y_1, \ldots, y_{2n})$. If Y is a second vector field in the Lie algebra of the Galois group, an easy computation shows that $[X, Y] = 0$. This proves the lemma. ■

Proposition 7.1. If a linear symplectic difference system is integrable then the neutral component of its Galois group is abelian. □

Proof. Invoking Lemma 5.2, we get that for every integer s the system $\phi Y = \phi^{s-1} A \cdots \phi A \cdot AY$ is symplectic and integrable. The lemma follows from this observation and from Lemma 5.3 together with Lemma 7.2 (indeed maintaining the notations of Lemma 5.3, Lemma 7.2 implies that G'^0 is commutative because G' is the Galois group of a symplectic and integrable system having a domain (R_i) as a Picard–Vessiot ring). ■

First proof of theorem 7.1. It is a direct consequence of Theorem 6.1 and Proposition 7.1. □

7.3 Second proof using Malgrange groupoid

Let $f : V \dashrightarrow V$ be a rational map and $\iota : \mathbb{P}^1(\mathbb{C}) \dashrightarrow V$ a ϕ-adapted curve.

Lemma 7.3. If the Malgrange groupoid of f is infinitesimally commutative then the Malgrange groupoid of $R_q f$ is infinitesimally commutative too. □

Proof. Differential invariants of f give differential invariants for $R_q f$ as described in Section 4.2. Because $R_q f$ is a prolongation, it preserves also the vector field D_j on $R_q V$: $(R_q f)^* D_j = D_j$. Hence Malgrange groupoid of $R_q f$ must preserve these vector fields. A direct computation shows that a vector field commutes with all the D_j if and only if it is a prolongation of a vector field on V. The compatibility of the prolongation with the Lie bracket proves the lemma. ■

Let $B_q\iota$ be the pull-back by ι of the frames bundle R_qV. The principal $GL_{2n}(\mathbb{C})$-bundle $B_1\iota$ is the bundle of frames of ι^*TV on $\mathbb{P}^1(\mathbb{C})$. Let $R_q\iota$ denote the extension of ι from $B_q\iota$ to R_qV. The system given by the restriction of $R_1 f$ to $B_1\iota$ is the fundamental discrete variational equation. Its solutions are the fundamental solutions of the variational equation. The following lemma follows from the definition of the discrete variational equation.

Lemma 7.4. The pull-back by $R_1\iota$ of differential invariants defined in a neighborhood of $\iota(\mathbb{P}^1(\mathbb{C}))$ are differential invariants of the fundamental variational equation. □

We will see in the second proof of Theorem 7.1 that some differential invariants are less easy to grab.

Lemma 7.5. If the difference system $\phi Y = AY$ on a trivial $SL_{2n}(\mathbb{C})$-principal bundle on $\mathbb{P}^1(\mathbb{C})$ with $A \in SL_{2n}(\mathbb{C}(z))$ is integrable then the fibers-tangent vector fields of the Lie algebra of its Malgrange groupoid commute. □

Proof. The proof follows exactly the proof of Theorem 4.1 ■

Our second proof of Theorem 7.1 relies on the following conjectural result:

Conjectural Statement: For linear (q-)difference systems, the action of Malgrange groupoid on the fibers gives the classical Galois groups.

The similar statement for linear differential equations is *true* and is proved in [11]. The proof of this conjectural statement in the q-difference case is a work in progress by A. Granier.

Second proof of theorem 7.1. Lemmas 7.3 and 7.4 prove the theorem when the first integrals are independent on a neighborhood of the invariant curve.

Let us first assume that the first integrals are defined on ι it, i.e. are in $\mathbb{C}[V]_\iota$ but not independent on the curve. Let H be a first integral defined in a neighborhood of the invariant curve and k be its generic valuation. The form dH is invariant and gives n differential invariants of order 1. If $k > 1$ then these invariants vanish above the invariant curve. The differentials of these invariants induce order 2 differential invariants. After k iteration of this process, one gets order k differential invariants but their restrictions on $B_k\iota$ give well-defined functions on $B_1\iota$. These functions are induced on $B_1\iota$ from the junior part H_ι° on ι^*TV.

By Ziglin lemma one can assume that the junior parts of the first integrals, hence the order k invariants induced are independent. One can apply Lemma 7.5 to get the infinitesimal commutativity of the vertical part of the Malgrange groupoid and, by the above conjectural statement, the almost commutativity of the difference Galois group.

Now let the first integral be any kind of rational functions on V. Such a H can be written $H = \frac{F}{G}$ with F and G in $\mathbb{C}[V]_\iota$. These two functions induced functions on the frame bundles, which are no more invariant. As in the previous case, the junior parts F° and G° of F and G are the restrictions of functions induced by dF and dG on the frame bundle. Let denote them by h_F^i and h_G^i for $1 \le i \le n$. Because H is invariant, F and G are semi-invariants: $F \circ f = aF$ and $G \circ f = aG$. Let us have a look at the lift of these functions on R_1V. One has $(h_F^j)(R_1 f) = h_{da}^j F + a(h_F^j)$. Let k be the generic valuation of F along ι; then, after k lift of the differential as functions on higher frame bundles, one gets functions h_F^α such that $h_F^\alpha(R_k f) = \cdots + a h_F^\alpha$ where the dots stand for terms vanishing above $\iota(\mathbb{P}^1(\mathbb{C}))$. The same computation on G shows that the junior part of H is an invariant of $R_1 f$ above ι. The arguments used to conclude the first case can be applied in this situation. This proves the theorem. □

8 Some Examples

In this section, we present some examples in the case $2n = 2$. In this situation, rational maps $f : V \dashrightarrow V$ are allowed to be nonsymplectic and integrability of f means existence of a nonconstant rational first integral.

8.1 Variants when $n = 1$

Following same lines as the proof of the main theorem, one gets.

Theorem 8.1. Let V be an algebraic variety of dimension 2 and consider a rational map $f : V \dashrightarrow V$ having a ϕ-adapted curve ι. If f gets a rational first integral then the intersection of the neutral component of the Galois group of the discrete variational equation of f along ι with $SL_2(\mathbb{C})$ is commutative. □

Proof. The proof follows the same lines as that of the main theorem (mores simple actually). ■

We also have the following useful result.

Theorem 8.2. Let V be an algebraic variety of dimension 2 and consider a rational function $f : V \dashrightarrow V$ having a ϕ-adapted curve ι. If one of the following conditions relative to the Galois group G of the discrete variational equation of f along ι holds then f is not integrable:

- $\mathrm{diag}(\mathbb{C}^*, \mathbb{C}^*)$ is conjugated to a subgroup of G;
- $\begin{pmatrix} 1 & \mathbb{C} \\ 0 & \mathbb{C}^* \end{pmatrix}$ is conjugated to a subgroup of G. □

Proof. Use the same arguments as for the proof of the main theorem and the fact that a rational function $F(z)(x, y)$ invariant by the natural action of the above groups (on (x, y)) is necessarily constant. ■

8.2 Example 1

Consider

$$f : \begin{array}{ccc} \mathbb{C}^2 & \dashrightarrow & \mathbb{C}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} & \mapsto & \begin{pmatrix} qx\frac{1}{1-\frac{y}{x-1}} \\ \underline{q}y\left(1-\frac{y}{x-1}\right) \end{pmatrix} \end{array}$$

where $q, \underline{q} \in \mathbb{C}^*$. The curve $\varphi(x) = (x, 0)$ is a σ_q-adapted to f. The variational equation of f along φ is given by

$$Y(qx) = \begin{pmatrix} q & q\frac{x}{x-1} \\ 0 & \underline{q} \end{pmatrix} Y(x). \tag{5}$$

If $\underline{q}$ is in general position then its Galois group is $\begin{pmatrix} 1 & \mathbb{C} \\ 0 & \mathbb{C}^* \end{pmatrix}$ so that f is not integrable. *A contrario*, remark that when $q\underline{q}$ is a root of the unity then the map f is integrable ($F(x, y) = (xy)^n$ is a first integral for $n \in \mathbb{N}$ large enough).

8.3 Example 2

Consider

$$f : \begin{array}{ccc} \mathbb{C}^2 & \dashrightarrow & \mathbb{C}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} & \mapsto & \begin{pmatrix} qx + ya(x) \\ yb(x) \end{pmatrix} \end{array}$$

with $b \neq 0$. The curve $\varphi(x) = (x, 0)$ is a σ_q-adapted to f. The variational equation of f along φ is given by

$$Y(qx) = \begin{pmatrix} q & a(x) \\ 0 & b(x) \end{pmatrix} Y(x). \tag{6}$$

Let us for instance study the case where $a(x) = b(x) = q\frac{x-1}{bx-c/q}$ with $b/c \notin q^{\mathbb{Z}}$. Remark that

$$\begin{pmatrix} q & q\frac{x-1}{bx-c/q} \\ 0 & q\frac{x-1}{bx-c/q} \end{pmatrix} = qz \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\frac{z-1}{bz-c/q} & \frac{(1+b)z-(1+c/q)}{bz-c/q} \end{pmatrix} z^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}^{-1},$$

so that (6) is rationally equivalent to (the system associated to) a basic hypergeometric equation with parameters $(1, b, c)$, the Galois group of which is $\left(\begin{smallmatrix} 1 & \mathbb{C} \\ 0 & \mathbb{C}^* \end{smallmatrix}\right)$ (see [20]). Consequently, f in not integrable.

8.4 **Example 3**

Consider

$$\begin{array}{rccc} f: & \mathbb{C}^2 & \dashrightarrow & \mathbb{C}^2 \\ & \begin{pmatrix} x \\ y \end{pmatrix} & \mapsto & \begin{pmatrix} ya(x) - x + 1 \\ xyb(y) \end{pmatrix} \end{array}$$

where $a \in \mathbb{C}(x)$ and $b \in \mathbb{C}(y)$ without pole in 0. The curve $\varphi(x) = (x, 0)$ is a τ_{-1}-adapted curve with $\tau_{-1}(z) = z - 1$. The variational equation of f along φ is given by

$$Y(1 - x) = \begin{pmatrix} -1 & a(x) \\ 0 & xb(0) \end{pmatrix} Y(x).$$

The algorithm exposed in [7] allows us to exhibit many nonintegrable cases.

8.5 **Example 4**

Consider

$$\begin{array}{rccc} f: & \mathbb{C}^2 & \dashrightarrow & \mathbb{C}^2 \\ & \begin{pmatrix} x \\ y \end{pmatrix} & \mapsto & \begin{pmatrix} qx + \tilde{q}y + (x-y)a(x, y) \\ \underline{q}x + \underline{\tilde{q}}y + (x-y)\underline{a}(x, y)) \end{pmatrix} \end{array}$$

where $a, \underline{a} \in \mathbb{C}(x)$ and $q, \tilde{q}, \underline{q}, \underline{\tilde{q}} \in \mathbb{C}$ are such that $\mathfrak{q} = q + \tilde{q} = \underline{q} + \underline{\tilde{q}} \in \mathbb{C}^*$. The curve $\varphi(x) = (x, x)$ is a $\sigma_{\mathfrak{q}}$-adapted to f. The variational equation of f along φ is given by

$$Y(\mathfrak{q}x) = \begin{pmatrix} q + a(x,x) & \tilde{q} - a(x,x) \\ \underline{q} + \underline{a}(x,x) & \underline{\tilde{q}} - \underline{a}(x,x) \end{pmatrix} Y(x). \tag{7}$$

This system is rationally equivalent to

$$Y(\mathfrak{q}x) = \begin{pmatrix} q - \underline{q} + a(x,x) - \underline{a}(x,x) & 0 \\ \underline{q} + \underline{a}(x,x) & \mathfrak{q} \end{pmatrix} Y(x). \tag{8}$$

One can see that generally f is not integrable.

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